Encoding Formula

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1 APOs

To recap, an anonymous piecewise object extends the definition of a piecewise object.

Namely, for a piecewise object ϕ,

$$\phi^\star = \begin{cases} \phi & \phi \text{ is defined} \\ \star & \phi \text{ is not defined} \end{cases}$$

This has infinitely many formulations, and on its own, does not restrict any of them. On its own, this is a mildly meaningless definition except for when dealing with (i.e. manipulating) the notation directly such as in polynomial interpolation. Instead, we can define several more objects to make this more readable.

Let μ be a solution to $\phi^\star$, that is, an object that satisfies $\phi^\star$:

$$\mu = \begin{cases} \phi & \phi \text{ is defined} \\ \star & \phi \text{ is not defined} \end{cases}$$

Therefore, we can write:

$$\phi^\star = \mu + \begin{cases} 0 & \phi \text{ is defined} \\ \star & \phi \text{ is not defined} \end{cases}$$

And define the latter anonymous piecewise object to be the *decider*, $[\phi]$ alongside some arbitrary object r. Hence:

$$\phi^\star = \mu + [\phi] \cdot r$$

The decider object, by its very nature, has a superposition property built-in. The above form is simple/single form, whereas we could alternatively write in extended form $\phi^\star$ as:

$$\phi^\star = \mu + \sum_n ([\phi]_n \cdot r_n)$$

Again, on its own this formula is virtually meaningless. However, once the form of the piecewise objects are determined, the formulation becomes more heavily restricted, as will be shown below.

1.1 A Note on Univariate Polynomial Interpolation

Polynomial interpolation is one of the ideas the APO framework was built around (this is not the only purpose of the existence of APOs, however). Mostly, standard interpolation talks about the lowest degree polynomial that satisfies an arbitrary number of points given (parametrically or otherwise).

Instead, we can derive the lowest degree polynomial using our standard APO/piecewise techniques; for example, $p(x)$. Then, we can write *all* polynomials that satisfy these points as:

$$P(x) = p(x) + [p(x)]r(x)$$

Where $[p(x)]$ are the roots of all points given. If points $\{(x_i, y_i) \mid i \in I\}$ are given, then this would be formulated as:

$$[p(x)] = \prod_{i \in I} (x - x_i)$$

And $r(x)$ is any arbitrary polynomial. $r(x) = 0$ gives, trivially, the lowest degree polynomial, $P(x)$. Notice that $\deg [p(x)] = |I|$ and $\deg p(x) = |I| - 1$ and hence if $r(x)$ is nonzero then $\deg P(x) \geq |I|$.

Moreover, while APOs cannot themselves rigourously prove that this gives all polynomials, both the fundamental theorem of algebra and factor theorem show that $P(x)$ gives all polynomials that satisfy these points.

By the factor theorem, $[p(x)]$ divides $P(x) - p(x)$; $P(x) - p(x) = k(x)[p(x)]$ for some $k(x)$. And since for all $P(x)$ there exists an $k(x) = r(x)$, we can write $P(x) = p(x) + [p(x)]r(x)$.

2 The Encoding Formula

The encoding formula, as shown below,

$$\phi^\star = \mu_1 + \sum_{k=1}^{n-1} (\mu_{k+1} - \mu_k) t_k$$

for objects $\mu_1, \mu_2, \ldots, \mu_n$ and arbitrary objects $t_1, t_2, \ldots, t_{n-1}$, can be derived using the definition of an anonymous piecewise object while attempting to encode multiple objects, functions or otherwise, into the same object. In other words, instead of fixing the decider object, we'll attempt to solve for the conditions under which each of the objects above all satisfy $\phi^\star$.

Namely, we take the extended form of an APO:

$$\phi^\star = \mu + \sum_{k=1}^{n-1} [\phi]_k t_k$$

And let μ_k and μ_{k+1} for all $k \in \{1, 2, \ldots, n-1\}$ be solutions (or objects which satisfy $\phi^\star$) such that:

$$\phi^\star = \mu_k + \sum_{k=1}^{n-1} [\phi]_k r_k \tag{1}$$

$$\phi^\star = \mu_{k+1} + \sum_{k=1}^{n-1} [\phi]_k s_k \tag{2}$$

Subtracting (1) $-$ (2) gives:

$$\begin{aligned} \mu_{k+1} - \mu_k &= \sum_{k=1}^{n-1} [\phi]_k (r_k - s_k) \\ &= \begin{cases} 0 & \phi \text{ is defined} \\ \star & \phi \text{ is not defined} \end{cases} \\ &:= [\phi]_k \end{aligned}$$

From here, we have our new decider objects definitions as chosen (this is not the only choice that can be made here; for example, one could let $[\phi]_k = \mu_{k+1} - \mu_1$). Since μ_1 satisfies $\phi^\star$ we can finally write:

$$\phi^\star = \mu_1 + \sum_{k=1}^{n-1} (\mu_{k+1} - \mu_k) t_k$$

This particular choice of formulation for ‘the’ encoding formula (although it’s inaccurate to say ‘the’ encoding formula since there are infinitely many choices for it) results in a nice telescoping pattern emerging.

Let $m < n \in \mathbb{Z}^+$ be a constant such that:

$$t_k = \begin{cases} 1 & k \leq m \\ 0 & k > m \end{cases}$$

Then $\phi^\star$ in fact telescopes to $\phi^\star = \mu_{m+1}$. Furthermore, our $\phi^\star$ object now takes the form, parameterised with m above:

$$\phi^\star = \begin{cases} \mu_1 & m = 0 \\ \mu_2 & m = 1 \\ \vdots & \vdots \\ \mu_n & m = n-1 \\ \star & \star \end{cases}$$

2.1 Vector Lines

Suppose we want to derive a parametric vector line equation; we let $\mu_k = \vec{v}_k$ for $k \in \{1, 2\}$.

Then we parameterise $\phi^\star = \vec{r}(t)$ (by $t \in \mathbb{R}$) such that we get the vector line equation:

$$\vec{r}(t) = \vec{v}_1 + (\vec{v}_2 - \vec{v}_1)t$$

By the encoding formula. We also notice that this takes the form of:

$$\vec{r}(t) = \begin{cases} \vec{v}_1 & t = 0 \\ \vec{v}_2 & t = 1 \\ \star & \star \end{cases}$$

Using our standard APO/piecewise techniques, this isn’t terribly difficult to derive by hand.

2.2 Vector Planes

Unlike the previous example, a parametric vector line, a parametric vector plane is far more difficult to derive using standard techniques used so far. So instead, given that a plane can be given by 3 vectors that span the plane, we let $\mu_k = \vec{v}_k$ for $k \in \{1, 2, 3\}$.

Then, we parameterise $\phi^\star = \vec{r}(u,v)$ (by $(u,v) \in \mathbb{R}^2$) such that we get the vector plane equation:

$$\vec{r}(u,v) = \vec{v}_1 + (\vec{v}_2 - \vec{v}_1)u + (\vec{v}_3 - \vec{v}_2)v$$

This has the piecewise form:

$$\vec{r}(u,v) = \begin{cases} \vec{v}_1 & (u,v) = (0,0) \\ \vec{v}_2 & (u,v) = (1,0) \\ \vec{v}_3 & (u,v) = (1,1) \\ \star & \star \end{cases}$$

2.3 Telescoping Property

A few of the derivations as below are a direct result of the telescoping property of the encoding formula; that is, when $t_k = 1$ for all k, the encoding formula reduces to a single term. This is not unique, in fact, the derivations as given below are moderately standard but are motivated in a different way. Realistically, the form

$$\phi^\star = \mu_1 + \sum_{k=1}^{n-1} (\mu_{k+1} - \mu_k) r_k$$

does not contain a telescoping property of sorts, but relies on $r_k = 1$ for some $k \in \{1, 2, \ldots, n-1\}$ and $r_k = 0$ everywhere else.

With this being said, however, similar formulas can still be derived using varying approaches. This demonstrates the flexibility of our encoding formula(s) and the importance of such. And so while our standard form presents itself in such a way we can reduce generally to:

$$\sum_{k=1}^{n-1} \mu_{k+1} - \mu_k = \mu_n - \mu_1$$

2.4 Geometric Series (Finite)

The closed forms for both finite and infinite geometric series are standard results in calculus, but what if we could derive them without calculus or writing out the series by hand? In fact, the approach we'll use is to encode each term/function, $\mu_1, \mu_2, \ldots, \mu_{n+1} = a, ar, \ldots, ar^n$ so that we have:

$$\phi^\star(r) = a + \sum_{k=0}^{n} (ar^{k+1} - ar^k) t_k$$

We then notice the following facts:

- $\phi^\star(r) = ar^{n+1} \iff t_k = 1$ by the telescoping property of the encoding formula.
- $ar^{k+1} - ar^k = (r-1)ar^k$

Using the latter fact, we can factor out the term $(r-1)$, and so we write:

$$ar^{n+1} = a + (r-1)\sum_{k=0}^{n} ar^k$$

Solving for the sum, we get that:

$$\sum_{k=0}^{n} ar^k = \frac{ar^{n+1} - a}{r-1}$$

In practice this almost seems tautological. Out of our formulation, we've already noticed that ar^{n+1} is the telescoping of all encoded functions, the monomials, and so the corresponding sum should telescope too. This also means we can go backwards too; that is, the geometric sum can be used to generate the original telescoping sum.

2.5 Geometric Series (Infinite)

The infinite geometric series derivation is incredibly similar to the finite geometric series'; instead, however, we'll let $n \to \infty$. Recall that $\mu_1, \mu_2, \ldots, \mu_{n+1} = a, ar, \ldots, ar^n$, and that:

- $\phi^\star(r) = ar^{n+1} \iff t_k = 1$ by the telescoping property.
- ar^{n+1} converges to 0 if and only if $-1 < r \leq 1$.
- $ar^{k+1} - ar^k = (r-1)ar^k$.

Using these facts, we'll write that:

$$0 = a + \sum_{k=0}^{\infty} (r-1)ar^k$$

Notice that for $r = 1$ the sum converges. However, factoring $r-1$ out of the sum will only work if the resultant sum itself converges, which only occurs for $|r| < 1$. Hence:

$$0 = a + (r-1)\sum_{k=0}^{\infty} ar^k$$

And solving for the series;

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$$

2.6 Recurrence Relations (Standard)

Suppose we want to solve for a specific sum or series:

$$\sum_{k=1}^{n} f(k)$$

Using our (telescoping; $r_k = 1$) encoding formula, we know that the summand is equal to $\mu_{k+1} - \mu_k$, which we'll call $F(k+1) - F(k)$. We also know, therefore, that $\mu_1 = F(1) := F_1$. Hence, in order to solve for the sum, we have to find some function $F(k)$ such that for all $k \in [1, 2, \ldots, n+1]$ we have:

$$\begin{aligned} F(k+1) - F(k) &= f(k) \\ F(1) &= F_1 \end{aligned}$$

Thus we write:

$$\phi^\star(n) = F_1 + \sum_{k=1}^{n} f(k)$$

And solving for the sum, using $\phi^\star(n) = F(n+1)$,

$$\sum_{k=1}^{n} f(k) = F(n+1) - F(1)$$

If you're familiar with the discrete/finite calculus, this is the equivalent of the discrete fundamental theorem of calculus. Interesting that this pops up here, but realistically, what our encoding formula (in the language of discrete calculus) ends up being is:

$$\phi^\star = \mu_1 + \sum_{k=1}^{n-1} (t_k \cdot \Delta\mu_k)$$

With this being said, most of the time it makes far more sense to try to find a sum indirectly, namely using a different formulation of μ_k that can be factored or otherwise manipulated. This is despite showing off the connection between encoding multiple objects into one and the discrete/finite calculus.

On a more fundamental and relevant level, this means what we're doing is encoding the function $F(n)$ in a single object for different discrete values; $F(1), F(2), \ldots, F(n+1)$. This is in essence multivariate interpolation, though, in order for telescoping to take effect, we set each of our 'parameters' to be 1. Furthermore, F has to be defined in terms of f and so we also reduce this to a recursion problem that, on the surface seems tautological, but yields a formulation of a given sum as shown above.

2.7 Cosine Sum in Arithmetic Progression

Inspiration/reference: How can we sum up sin and cos series when the angles are in arithmetic progression? It's here we take inspiration from the trigonometric identity:

$$\sin(a) - \sin(b) = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$

Briefly, this can be derived using trigonometric identities $\sin(p+q)$ and $\sin(p-q)$, using the substitution $a = p+q$ and $b = p-q$.

In order to come up with a closed form for this sum, we know for our lower/upper bounds to be 0 and n respectively, we let $k \in \{0, 1, \ldots, n+1\}$

and $\mu_k(x) = \sin(x + dk)$ for some d. Furthermore, to be able to manipulate the series, we let $t_k(x) = 1$ in our encoding formula. Therefore:

$$\phi^\star(x) = \sin(x) + \sum_{k=0}^{n} (\sin(x + kd + d) - \sin(x + kd))$$

Noting that $\phi^\star(x)$ denotes the $n+1$th case in this instance, we have $\phi^\star(x) = \sin(x + d(n+1))$. Furthermore, we apply our trigonometric identity to the sum and factor out, giving us:

$$\sin(x + d(n+1)) = \sin(x) + 2\sin\left(\frac{d}{2}\right)\sum_{k=0}^{n}\cos\left(x + kd + \frac{d}{2}\right)$$

Letting $x \mapsto x - \frac{d}{2}$, and rearranging, we arrive at:

$$\sum_{k=0}^{n}\cos(x + dk) = \frac{1}{2}\csc\left(\frac{d}{2}\right)\left(\sin\left(x + \frac{(2n+1)d}{2}\right) - \sin\left(x - \frac{d}{2}\right)\right)$$

Applying the same identity as before,

$$\sum_{k=0}^{n}\cos(x + dk) = \frac{\cos\left(x + \frac{nd}{2}\right)\sin\left(\frac{(n+1)d}{2}\right)}{\sin\left(\frac{d}{2}\right)}$$

2.8 Sine Sum in Arithmetic Progression

This derivation is quite similar to the previous, but we instead rely on the trigonometric identity (which can be derived nearly identically)

$$\cos(a) - \cos(b) = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$

Again, to set up our closed form, we know for our bounds to be 0 and n, we let $k \in \{0, 1, \ldots, n+1\}$ for $\mu_k = \cos(x + dk)$, for some d. To finalise setup, we allow $t_k(x) = 1$ for all k in the encoding formula, such that we have:

$$\phi^\star(x) = \cos(x) + \sum_{k=0}^{n} (\cos(x + kd + d) - \cos(x + kd))$$

Again $\phi^\star(x)$ represents the $n+1$th case and so $\phi^\star(x) = \cos(x + (n+1)d)$. Applying the identity and again factoring, we have:

$$\cos(x + d(n+1)) = \cos(x) - 2\sin\left(\frac{d}{2}\right)\sum_{k=0}^{n}\sin\left(x + kd + \frac{d}{2}\right)$$

Rearranging for the sum, and letting $x \mapsto x - \frac{d}{2}$, we have:

$$\sum_{k=0}^{n}\sin(x + kd) = \frac{1}{2}\csc\left(\frac{d}{2}\right)\left(\cos\left(x - \frac{d}{2}\right) - \cos\left(x + \frac{(2n+1)d}{2}\right)\right)$$

Rewriting using our identity as above;

$$\sum_{k=0}^{n}\sin(x + kd) = \frac{\sin\left(x + \frac{nd}{2}\right)\sin\left(\frac{(n+1)d}{2}\right)}{\sin\left(\frac{d}{2}\right)}$$

2.9 Maclaurin Series

For the sake of this section, we'll assume convergence happens nicely and that we can, in fact, take the Maclaurin series of a function.

The idea of encoding $\mu_1, \mu_2, \ldots, \mu_n = 1, x, \ldots, x^n$ as $n \to \infty$ gives us a (factored) polynomial series in general:

$$\phi^\star(x) = 1 + (x-1)\sum_{k=0}^{\infty} t_k(x)x^k$$

For some parameters t_k. Up until now, we've abused the telescoping property for when $t_k = 1$. But what if, instead, we pick some alternative $t_k(x)$; namely:

$$t_k(x) = \frac{f^{(k)}(x)}{k!}$$

For some function $f(x)$. Then,

$$\phi^\star(x) = 1 + (x-1)\sum_{k=0}^{\infty} \frac{f^{(k)}(x)}{k!}x^k$$

We notice immediately that this is the Maclaurin series of $f(x)$ provided that it converges;

$$\phi^\star(x) = 1 + (x-1)f(x)$$

This is helpful because one might be able to, instead of picking $f(x)$, pick $\phi^\star(x)$ and work backwards, thereby encoding a different set of μ in order to get a desired $\phi^\star(x)$.

Interestingly, if we have $\phi^\star(x) = 0$ then $f(x) = \frac{1}{1-x}$. Equivalently, if we have the coefficients of the Maclaurin series equal to 1, then for $n \in \mathbb{Z}^+ \cup \{0\}$ $f^{(n)}(x) = n!$; this gives the series for $\frac{1}{1-x}$ also since $x^n \to 0$ as $n \to \infty$ when it converges (hence equivalence).

2.10 Deparameterising and Gluing

When using the encoding formula in the context of encoding functions, i.e. we have a set of functions $\mu_1(x), \mu_2(x), \ldots, \mu_n(x)$ we're left with a set of parameters that also can be dependent on x, namely, $t_1(x), t_2(x), \ldots, t_{n-1}(x)$ such that:

$$\phi^\star(x) = \mu_1(x) + \sum_{k=1}^{n-1} (\mu_{k+1}(x) - \mu_k(x))t_k(x)$$

The only is now, though, that we have the extra set of parameters that we're left to deal with. Turns out, what if we choose these terms to be dependent on the value of x itself somehow?

We know already that our standard form is designed to telescope based on when $t_k(x) = 1$. Another notation we're familiar with does the same thing: Iverson brackets, which are defined in this context as follows, for some statement $S(x)$:

$$[S(x)] = \begin{cases} 1 & S(x) \\ 0 & \neg S(x) \end{cases}$$

Namely, Iverson brackets behave in the way we want when we describe $t_k(x) = [x \geq x_k]$: for $x_1 \leq x_2 \leq \cdots \leq x_{n-1}$ we have that for $x \geq x_k$, $t_i(x) = 1$ for $i \in \{1, 2, \ldots, k\}$ and so the telescoping effect occurs.

Thus, we're left with the following gluing formula:

$$\phi^\star(x) = \mu_1(x) + \sum_{k=1}^{n-1} (\mu_{k+1}(x) - \mu_k(x))[x \geq x_k]$$

This is only one gluing formula; for example, one can instead opt to go backwards with $[x \leq x_k]$.

Furthermore $\phi^\star(x)$ is continuous iff $\mu_k(x)$ is continuous for all k, and that $\mu_{k+1}(x_k) = \mu_k(x_k)$. This continuity property allows us to simplify our expression:

$$\begin{aligned}
(\mu_{k+1}(x) - \mu_k(x))[x \geq x_k] &= \begin{cases} \mu_{k+1}(x) - \mu_k(x) & x \geq x_k \\ 0 & x \leq x_k \end{cases} \\
&= \begin{cases} \mu_{k+1}(x) - \mu_k(x) & x \geq x_k \\ \mu_{k+1}(x_k) - \mu_k(x_k) & x \leq x_k \end{cases} \\
&= \mu_{k+1}\left(\begin{cases} x & x \geq x_k \\ x_k & x \leq x_k \end{cases}\right) - \mu_k\left(\begin{cases} x & x \geq x_k \\ x_k & x \leq x_k \end{cases}\right) \\
&= \mu_{k+1}(\max(x, x_k)) - \mu_k(\max(x, x_k))
\end{aligned}$$

And therefore, for continuous $\phi^\star(x)$ we have:

$$\phi^\star(x) = \mu_1(x) + \sum_{k=1}^{n-1} (\mu_{k+1}(\max(x, x_k)) - \mu_k(\max(x, x_k)))$$

In general, though, rather than relying on this formula, it's better to create it when attempting to derive another thing. The reason being that there may be a far simpler formulation for what you're trying to achieve.

2.11 Functions and Relations in R^2

Suppose that we have some relations $\mu_1(x, y) = 0, \mu_2(x, y) = 0, \ldots, \mu_n(x, y) = 0$ and, like in each other example, we wish to encode these somehow in a way that is also a relation; $\phi^\star(x, y) = 0$.

The quick and simple way is to plug these straight into the encoding formula, but if we do this, we lose parameterisation of a function. For consistency and functional purposes, this is not beneficial. So instead, we also add the zero function in "front" of $\mu_1(x, y)$. This way, we have:

$$\phi^\star(x, y) = \mu_1(x, y)t_1(x, y) + \sum_{k=1}^{n-1} (\mu_{k+1}(x, y) - \mu_k(x, y))t_k(x, y) = 0$$

From here, it's absolutely not necessary to deparameterise. However, there are a few main ideas I'd go through:

- All relations at once

- One relation at a time using a single variable

What we mean by "all relations at once" is we mean to say that we construct a 3 dimensional surface, $z = \phi^\star(x, y)$, for which $z = 0$ yields the given relations. It turns out there already exists a formulation for this known as the null factor law; $\mu_1(x, y)\mu_2(x, y)\dots\mu_n(x, y) = 0$. In terms of our formulation, this is fairly easy to represent:

$$t_1(x, y) = t_2(x, y) = \cdots = t_{n-1}(x, y) = \prod_{k=1}^{n-1} \mu_k(x, y)$$

Which gives:

$$\phi^\star(x, y) = t_1(x, y) \left[\mu_1(x, y) + \sum_{k=1}^{n-1} (\mu_{k+1}(x, y) - \mu_k(x, y)) \right] = 0$$

By the telescoping property and the representation as above, we subsequently get:

$$\phi^\star(x) = \prod_{k=1}^{n} \mu_k(x, y) = 0$$

The latter idea, representing a single function/relation at a time using a single variable, is far more difficult. Reparameterising $\mu_k(x, y) = \mu_k(t)$ for $t \in \mathbb{R}$, we let:

$$(t_1, t_2 \dots, t_{n-1})(t) = \begin{cases} (1, 0, 0, 0, \dots, 0) & t = 0 \\ (1, 1, 0, 0, \dots, 0) & t = 1 \\ (1, 1, 1, 0, \dots, 0) & t = 2 \\ \vdots & \vdots \\ (1, 1, 1, 1, \dots, 1) & t = n - 2 \\ \star & \star \end{cases}$$

An easy-to-observe solution to this problem is:

$$(t_1, t_2, \dots, t_{n-1})(t) = ([t \geq 0], [t \geq 1], \dots, [t \geq n - 1])$$

Which again makes use of the Iverson bracket notation. However, it renders the solution non-elementary and inherently piecewise-defined (as well as non-smooth). The alternative, of course, is viewing the problem as an interpolation problem and interpolating in any which way is desired to suit the requirements at the time.